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A GENERALIZED CONCEPTION OF AREA: APPLICATIONS TO COLLINEATIONS IN THE PLANE

BY EDWIN BIDWELL WILSON

IN a paper presented to the summer meeting, 1902, of the Deutsche Mathematiker-Vereinigung and to be published in the *Jahresbericht* of that society, I pointed out the way in which one might arrive at a conception and numerical measure of volume in a linear space of any number of dimensions without the use of the conception or measure of length. The object of the present article is to discuss from a different point of view this same problem for a space of two dimensions, the plane; and to give some applications of the new notion of area thus obtained to results already known and to some that it is hoped may be new.

The reasons which render justifiable some change in the conception of area become apparent on examining the most evident properties of the successive subgroups of the general projective group. The general group G has eight degrees of freedom. There is a subgroup A generally known as the affine group from the fact that it leaves the line at infinity fixed as a whole though admitting a redistribution of the individual points upon that line. This group has six degrees of freedom. This freedom is cut down to five degrees in passing to the next subgroup L which possesses the characteristic property of conservation of area and which will occupy much attention in the following pages. Should one desire to introduce length it becomes necessary, if the language of non-euclidean geometry be used for a moment, to fix a system of two points, ordinarily the circular points, upon the line at infinity. The subgroup of L which is thus obtained is the group R of rigid motions; provided these be understood to include changes of symmetry in addition to the truly mechanical motions. This group R has three degrees of freedom only. To study the properties of the group L as much as possible by means of its characteristic invariant, *i. e.* area, and with a minimum reference to the more special invariant, *i. e.* length, of the subgroup R may be a worthy object in itself and may be expected in some small way to lead to valuable points of view as in a much larger way the use of the cross-ratio in treating the general group G is more suggestive than the direct employment of length.

1. Equivalence of area and its generalization. In dealing with area the triangle is taken as the element. If the vertex of a triangle be displaced parallel to the base, the area understood in the ordinary manner is unchanged. By displacing one vertex parallel to its opposite side, then another parallel to the opposite side of the triangle thus obtained, and by continuing this process of displacement, any given triangle ABC may be carried over into a triangle $A_1B_1C_1$ quite different. It is not difficult to grant that by properly carrying out the construction any triangle ABC may be transferred into any other triangle $A'B'C'$, provided only that the areas of the two triangles have the same magnitude and the same sign, that is the same cyclic order of the letters.

The idea therefore which enters necessarily into the conception of equivalence of triangles and which at the same time is independent of the concept of length is the idea of parallel lines and only that. Acting on this clew the following definition for the generalized equivalence of triangular areas will be laid down.

Definition: Two triangles ABC and $A'B'C'$, neither of which cuts the line l or touches it, are said to be equivalent in area *with respect to that line* l when and only when one triangle may be transformed into the other by displacing a vertex along the line joining that vertex to the intersection of the opposite side with the line l and by repeating this operation successively, upon the various intermediary triangles thus formed, a finite number of times. (See figure 1.)

A glance at the figure shows that the successive triangles in the sequence are ABC, ABC', $A'BC'$, $A'B'C'$, and that they preserve the same cyclic order of the letters. The line l in the conception of generalized equivalence plays the same rôle as the line at infinity in the ordinary conception. It is also evident that if $A'B'C'$ is equivalent to ABC, then inversely ABC is equivalent to $A'B'C'$; for the construction can be carried out indifferently backward or forward. For similar reasons the statement that two triangles equivalent to the same triangle are equivalent to each other becomes obvious.

As a matter of definition two triangles which have the same base and whose two opposite vertices are collinear with the point of intersection of the base and the line l are equivalent. And now, if that definition is to be useful, it is necessary that conversely two triangles ABC and ABC' which have the same base AB and whose vertices are not collinear with the intersection of AB and l shall be non-equivalent. From another standpoint this is merely the demand that the axiom "the whole is not equivalent to its part" shall hold for our generalization of the conception of area. For it is possible to construct a triangle $AB\overline{C}$ which shall be equivalent to ABC' and hence to ABC and such that the point $\overline{C}$ lies upon the line AC. If the point $\overline{C}$ does not coincide with C one of the triangles ABC and $AB\overline{C}$ must contain the other and the whole would be equivalent to its part. It becomes necessary to prove that no such case can arise out of the definition.

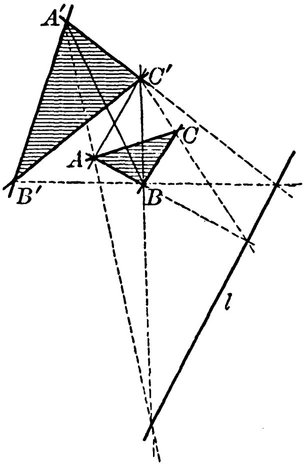

FIG. 1.

To give a geometrical proof apparently offers grave difficulties owing to the large number of steps which may be inserted between any initial triangle and its final equivalent. In the paper referred to above I gave the proof analytically, without in anyway introducing the conception or measure of length, by employing a system of projective coordinates dependent upon the projective definition of a number-system, which in turn is deduced from harmonic constructions. Although the germ of this number-system and system of coordinates may possibly be seen lurking in the geometrical nets of Hamilton and Möbius, yet the first to give them even a partially satisfactory development was von Staudt in his *Beiträge zur Geometrie der Lage.* The treatment has been amplified and bettered by his followers Klein*, Killing†, Pasch‡, Schur§. These matters are however somewhat recondite. Here we shall follow the plan adopted frequently in the presentation of synthetic projective geometry and practically always in that of analytic projective geometry. We shall have recourse to the ordinary conceptions of area, for convincing ourselves of the truth of some fundamental properties of the generalized area.

For example, if in the case under discussion it were possible to carry the triangle ABC into $AB\overline{C}$ as explained above, then if the line l be projected to infinity, it would be possible by parallel line constructions to carry a triangle $A_1B_1C_1$, the projection of ABC, into $A_1B_1\overline{C}_1$, the projection of $AB\overline{C}$. This is an absurdity inasmuch as the triangles $A_1B_1C_1$ and $A_1B_1\overline{C}_1$ cannot have the same area in the ordinary sense of the word. Thus the difficulty of the part being equivalent to the whole is disposed of even in the generalized notion of area with respect to a line l.

Theorem 1. A triangle ABC is equivalent to each of the triangles CAB, BCA obtained by permuting the letters cyclically.

Draw through B and C respectively lines cutting the line l in its intersections with AB and AC. Let these lines intersect in C'. The following sequence of equivalent triangles demonstrates the equivalence between ABC and CAB:

$$ABC = ABC' = CBC' = CAC' = CAB.$$

The equivalence between ABC and BCA is proved in the same manner.

2. Non-equivalence of generalized area. For the immediate present to simplify matters it will be supposed that the line l to which area is referred is in a very remote region of the plane or even at infinity. This restriction will be considered and removed later where the questions connected more subtly with the ideas of the sign and infinite magnitude of areas will be more easily handled than at present.

Definition. The area of a triangle is said to be reversed in sign when the cyclic order of the letters is changed; *i. e.*, $ABC = -ACB = -CBA = -BAC$. In general two triangles ABC and $A'B'C'$ are said to be equivalent negatively when one may be carried into the other with the cyclic order of the letters reversed.

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* *Math. Annalen*, vol. 37; *Nicht-Euclidische Geometrie* (autographierte Vorlesungshefte), vol. 2.

† *Grundlagen der Geometrie*, vol. 2, pp. 107 et seq., pp. 119 et seq.

‡ *Neuere Geometrie.*

§ *Math. Annalen*, vol. 53.

THEOREM II. Two triangles equivalent negatively to a third are equivalent (positively) to each other.

THEOREM III. The necessary and sufficient condition that two triangles ABC and ABC' upon the same base AB shall be equivalent negatively is that the line CC' shall be divided harmonically by AB and the line of reference l.

First, if the triangles are negatively equivalent displace the vertex $\bar{C}'$ along the line joining C' to M, the intersection of l and AB, until it falls upon the point $\bar{C}'$ such that $A\bar{C}'$ and CB meet on l. Join $C\bar{C}'$. Then

$$-ABC' = AC'B = A\bar{C}'B = A\bar{C}'C.$$

By supposition ABC' and ABC are negatively equivalent. Hence $A\bar{C}'C$ is equivalent to ABC. Hence BC' and AC meet on l. $AB\bar{C}C'$ is therefore a complete quadrangle. The opposite sides $C\bar{C}'$ and AB are divided harmonically by the diagonal l. The pencil composed of AB, l, MC, $M\bar{C}'$ is harmonic and the transversal CC' is divided harmonically by AB and l. The necessity of the condition announced in the theorem has thus been shown. The sufficiency may be proved in a converse manner.

As a corollary of Theorems II and III it follows that if two triangles ABC, ACD have the same vertex, a common side AC, and bases in the same straight line BCD, the necessary and sufficient condition that they be equal is that BD be divided harmonically by C and the line l.

If moreover upon the base produced of a given triangle ABC we construct the fourth harmonic D of B with respect to C and l, and then the fourth harmonic E of C with respect to D and l, and so on, the series of triangles

$$ABC, \quad ACD, \quad ADE, \quad \cdots$$

will all be mutually equivalent. And it may be mentioned that the points $B, C, D, E, \cdots$ to the point M in which BC cuts l, are the points which may be taken to correspond to the integers $0, 1, 2, 3, \ldots, \infty$ in the projective number system above referred to.

It may also be seen that the area ABM must needs be infinite if the area ABC be considered other than zero. As it would be unnatural to denominate any two-dimensional triangle ABC of zero area, the result follows that any triangle MNP which has one vertex P upon the line l is of infinite area.

For, an arbitrary triangle MNQ, where Q lies upon NP, may be taken as of unit area and the triangle MNP can not then be divided into a finite number of triangles equivalent to MNQ. As an extension, any triangle MNP which cuts the line l in any manner cannot be finite in area with respect to that line l. The reason therefore appears, as perhaps it has not yet appeared, why the restricting clause " neither of which cuts the line l or touches it" was inserted in the definition of equivalence.

To divide a triangle ABN which does not cut the line l into n equivalent parts by lines through the vertex A is an operation dependent upon the system of integers just mentioned. Choose at random any line q passing through B and not coincident with BC. Choose arbitrarily a point C_1 corresponding to the number 1. On the line q construct, as above described, the integers up to n, using the point in which q cuts l as the point ∞ of the line q. Let these integers be represented by C_1, D_1, . . . N_1. Let NN_1 cut the line l in the point O. Let OC_1, OD_1 . . . cut BN in C, D, . . . respectively. The triangles ABC, ACD, . . . are then obviously equivalent and each may be regarded as the nth part of ABN.

The areas of two triangles ABC and $A'B'C'$ may be compared to any degree of numerical approximation by a method well known in elementary geometry. The triangle ABC may be divided into n equivalent parts of which one is ABB_1. Upon $A'B'$ a triangle $A'B'B'_1$ may be constructed equivalent to ABB_1. The triangle $A'B'C'$ may then be filled up with k triangles $A'B'B'_1$, $A'B'_1B'_2$, $A'B'_2B'_3$, . . ., $A'B'_{k-1}B'_k$, which are equivalent, and one triangle $A'B'_kC'$, which is smaller. The ratio of ABC to $A'B'C'$ is n to k approximately—the approximation being less than one part in n. If the two triangles are commensurable two numbers n, k may be found such that the ratio is exactly n to k with no remainder. If the triangles are incommensurable the ratio n to k, as n is taken indefinitely greater and greater, will approach an irrational number as a limit.

A definite triangle $U_1U_2U_3$ may be assumed arbitrarily as a triangle of unit area with respect to the line l. By methods of comparison explained above, the numerical value of the area of any triangle ABC relatively to $U_1U_2U_3$ may be determined. As in the ordinary conception of area, the area of a figure not triangular must be determined by dividing that area into a finite or infinite number of triangles.

Thus the first object of this paper, namely, to explain the ideas of area with respect to a line and of the numerical measure of that area has been accomplished. There remains only the removal of the restriction imposed at the beginning of this section.

Consider any three lines a, b, c which do not all pass through the same point. These three lines divide the plane into four distinct regions such that it is possible to pass from any point of one region to any other point of that region without crossing any of the three lines, although crossing the line at infinity may be necessary. To pass from one region of the plane to another it is necessary to cross one side of the triangle. A line l which does not pass through any of the three points of intersection of the lines a, b, c cuts across three and only three of these four regions. There remains always one region, and only one, which has no point in common with l and consequently is a triangle not excluded from consideration by the definition of equivalence. If two such triangles are given their relative signs might give difficulty. The fact, too, that one or both of these triangles may be of what we should ordinarily regard as infinite extent is the only other difficulty. This latter has already been spoken of. The former may be removed by considering the sense (positive or negative) of the ranges in which the sides of the triangle cut the line of reference l. For example let the triangles be given by the lines a, b, c and a', b', c'. Let a, b, c cut l in A, B, C respectively; and let a', b', c' cut it in A', B', C'. The triangles will be said to have the same sign or opposite signs relatively to l according as the ranges ABC and $A'B'C'$ upon l have the same or opposite sense. Hence evidently two triangles which lie wholly on the same side of l have the same sign if their vertices follow in the same cyclic order; two triangles which lie wholly on opposite sides of l have the same sign if their vertices follow in opposite cyclic orders. For triangles which lie partly on one side, partly on the other,— that is, for triangles which cross the line at infinity,— no such simple rule can be stated. It is only safe to consult the senses of the ranges which the sides cut out on l.

3. Constructions and miscellaneous applications. There will occur so many occasions in the course of this section to speak of two lines which have the property of meeting on the line l with respect to which area is measured, that some special designation will be convenient. Two such lines will be said to be *parallel with respect to l*. It will generally suffice to abbreviate the expression to simply *parallel*, leaving the reference to the line l to be understood; for the line at infinity and parallelism in the elementary sense of the word will not, one may almost say cannot, be of importance in the subject under discussion here.

Construction. To construct a triangle upon a given base $A'B'$ which shall be equivalent to a given triangle ABC. (See figure 2.)

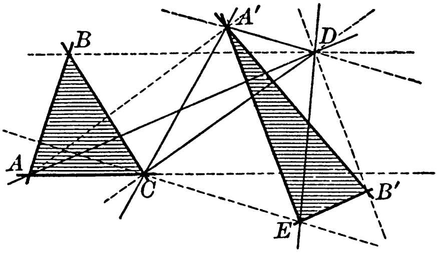

FIG. 2.
In this figure the line l has been taken at infinity for convenience in drawing.

Join AA'. From C draw a line parallel (in the extended sense) to AA'; and from B a line parallel to AC. Let them intersect in D. Join DB' and DA'. From A' draw a line parallel to DB'; and from C a line parallel to $A'D$. Let them intersect in E. Then $ABC=ADC=A'DC=A'DE=A'B'E$. The number of steps required to pass from one triangle to another is thus seen to be four. The construction fails when D cannot be found except on the line l; that is, when A' lies on the side AC. As A' can not lie on all three of the sides AC, CB, BA, it is always possible to carry the triangle ABC into a triangle upon the base $A'B'$, though that side $A'B'$ may correspond to AC or CB instead of to AB. Of course it must be remembered that A' and B' can not be situated on the line l and further that the segment $A'B'$ is that portion of the whole line $A'B'$ which does not cut l. Figure 3 shows an example of this case. In the succession of triangles $A'DE$ is not shaded owing to the confusion it would introduce with $A'B'E$. If the construction is performed in the inverse order the triangle $A'B'E$ will be carried over into ABC. Thus there is obtained an example of the construction of a triangle equivalent to a given triangle which lies on two sides of l but does not cut l.

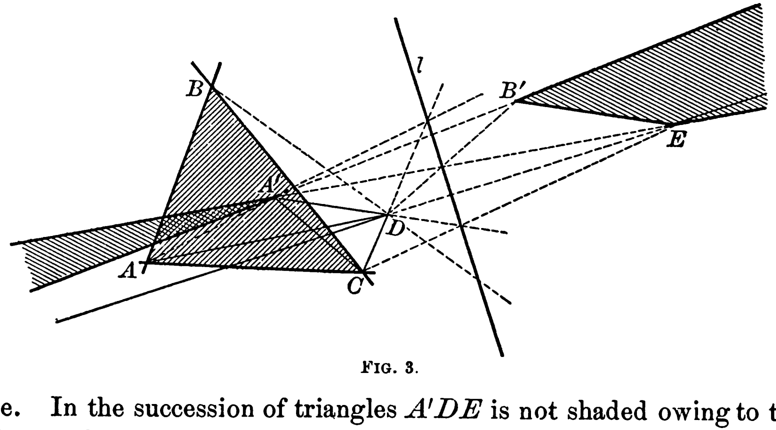

FIG. 3.

Theorem IV. The envelope of a line which cuts off a constant area from two given fixed lines is a conic tangent to those two lines at their points of intersection with the line of reference l. (See figure 4.)

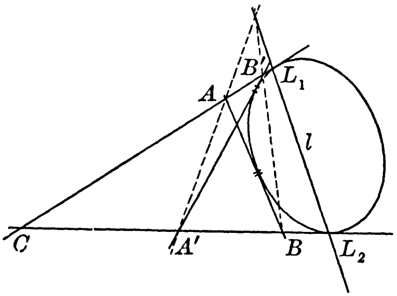

FIG. 4.

Let ACB and $B'CA'$ be two positions of the triangle. Join $A'A$ and BB'. As the areas of the triangles are equal the differences between those areas and the common area ACA' are equal. Therefore

$$AA'B' = AA'B.$$

The vertices B and B' must lie on a line passing through the intersection of the common base $A'A$ and the line l. If then the points of intersection of l with the fixed lines be L_1 and L_2, the ranges

$$CA'BL_2 \text{ and } CAB'L_1$$

are perspective. By interchanging the position of the points in both point-pairs of a given range the resulting range is projective to the given range. Hence

$$CAB'L_2 \barwedge L_1B'AC.$$

Hence

$$CA'BL_2 \doublebarwedge CAB'L_1 \barwedge L_1B'AC.$$

The lines joining corresponding points of two projective ranges envelope a conic which is tangent to the lines on which the ranges lie at the points which correspond to the point of intersection of the two ranges. Hence $A'B'$ and AB touch a conic tangent to the lines CL_1 and CL_2 at L_1 and L_2.

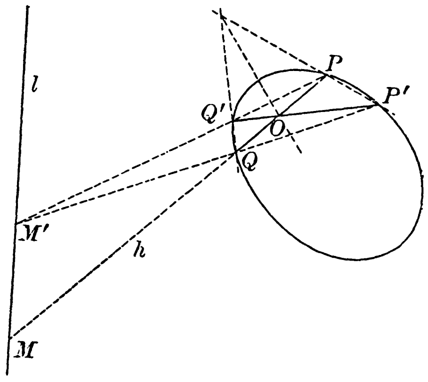

FIG. 5.

Theorem V. A line h moves so as to cut off a constant area from a given curve which has only two real intersections with any line (the curve is oval). Then h is divided harmonically with respect to the given curve by the point of intersection of h and l and by the point of tangency of h to its envelope. (See figure 5.)

Let PQ and $P'Q'$ be two positions of h. Let $P'Q'$ be considered as approaching PQ as a limit. Let PQ and $P'Q'$ intersect in O. In the limit O is the point of tangency of PQ with its envelope. The sectors PQP', $Q'QP$ are equal. The small curved segments upon the bases PP' and QQ' are insignificant compared with these sectors. Hence the triangles PQP' and $Q'QP$ are practically equal: the limit of their ratio is unity. In the limit therefore the lines PQ' and $P'Q$ meet on l in a point M. By the properties of the complete quadrangle $PP'QQ'$ the points O and M are separated harmonically by the curve. In the figure the point M' has been placed upon the line l to suit the requirements of Theorem VI. In general M' when approaching M will not move along the line l.

This proof of this theorem possesses the same degree of unsatisfactoriness as that of the corresponding theorem in the ordinary treatment of areas. There are numerous points to be filled in, some restrictions to be placed upon the given curve, etc. These difficulties are involved rather in the nature of the subject treated than in the particular method now employed and hence will be waived. By making use of this theorem and those theorems concerning conic sections which are most fundamental from von Staudt's point of view* it is easy to prove the following

Corollary. The envelope of a chord which cuts off a constant area with respect to a line l from a given conic K is a conic K' which is tangent to K in two points, real or imaginary — the chord of contact being the line l. Or, to avoid the possible imaginary case, the statement may be made thus: the polarities which define K and K' set up the same involution upon the line l. In case the line l cuts K the involution is hyperbolic and the points of contact of K and K' are real; in case l does not cut K the involution is elliptic and the points of contact are no longer real; in the intermediate case where l is tangent to K, K and K' have contact of the third order at the point of tangency.

THEOREM VI. The necessary and sufficient condition that a chord PQ cut off a constant area from a conic K is that the range of points P upon the conic shall be projective to the range of points Q upon the conic and that the axis of the projectivity of these two ranges of the second order shall be the line of reference l. (The axis of a projectivity is the line which is the locus of the points of intersection of the lines joining non-corresponding points of any two corresponding pairs of points in the projectivity.)

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* See a "Note on the Geometrical Treatment of Conics," by C. A. Scott, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900–01), p. 64.

This theorem is merely another statement of the corollary above. The proof may be given like that of Theorem V. But the most serious difficulties in the theory of limits here vanish of themselves because the point M' of intersection of $P'Q$ and PQ' lies necessarily upon l, the axis of projectivity. Hence the triangles PQP' and $QP'Q'$ are rigorously equal. The demonstration of Theorem VI should therefore be given independently of the more or less unsatisfactorily demonstrated Theorem V.

If two triangles PAB and $PA'B'$ which have the same vertex P and whose bases AB and $A'B'$ lie upon the same line are to be equivalent it is evidently necessary that the segments AB and $A'B'$ shall lie either partly or wholly without one another. Otherwise one triangle would be merely a part of the other. Suppose that the first case obtains. Let the order of the points upon the base line be $AA'BB'$. Then the triangle $PA'B$ is common to the two given triangles and hence PAA' is equivalent to PBB'. The question of the equivalence of two triangles with common vertex and collinear bases, therefore, reduces in every case to an equivalence in which the bases do not overlap.

THEOREM VII. The necessary and sufficient condition that two triangles PAB and $PA'B'$, whose bases lie upon the same line but do not overlap, shall be equivalent with respect to a line l not cutting either of them, is that the line l shall pass through the external point of harmonic division of the pairs of points AB' and $A'B$.*

Let O_1 be the internal point of division and O_2 the external point. Then by Theorem III,

$$PBO_1 = - PA'O_1 \text{ and } PAO_1 = - PB'O_1$$

with respect to any line l through O_2. By subtraction

$$PAB = - PB'A' = PA'B'.$$

This proof may be given very neatly without reference to the negative areas upon which Theorem III depends. The construction is as follows. Produce AP until it cuts l in A_1. Draw BA_1 and let it cut PB in C. The triangles PAB and PAC are equivalent. In like manner produce $B'P$ until it cuts l in B_1'. Draw $A'B_1'$ and let it cut PA in C'. Then the triangles $PA'B'$ and

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* By the external point will naturally be meant that one through which a line l may be drawn not cutting either of the triangles. Through the other point of harmonic division no line could be drawn not cutting either PAB or $PA'B'$.

$PC'B'$ are equivalent. Let $B'C'$ and AC intersect in H_1; $A'C'$ and BC, in H_2. Owing to the properties of the complete quadrilateral it is evident that PH_1H_2 is a straight line passing through O_1 and that CC' passes through O_2. The ranges

$$AC'PA_1 \doublebarwedge B'CPB'_1 \text{ with respect to } O_2.$$

Hence

$$AC'PA_1 \barwedge CB'B'_1P.$$

The triangles PAC and $PC'B'$ are therefore equal with respect to l by Theorem IV.

4. Application to the point-line involutory collineation. The sole involutory collineation in the plane is the point-line reflection. By this is meant the transformation in which a point P is replaced by a point P' such that the line PP' passes always through a fixed point O called the center and is divided harmonically by O and a fixed line p called the axis. This transformation is a projective generalization of ordinary reflection in a mirror. For this reason it will often be found convenient to use the words *projective reflection* to stand for the generalized transformation.

THEOREM VIII. If a triangle ABC is transformed into $A'B'C'$ by a projective reflection the areas of the triangles ABC and $A'B'C'$ are equal with reference to the axis of the reflection.

Consider first a triangle AOB and the corresponding triangle $A'OB'$. Let OAA' cut the axis l (or p) in A'', and OBB' in B''. Then the ranges

$$AOA'A'' \quad \text{and} \quad BOB'B''$$

are harmonic. But one pair of points in harmonic range may be interchanged without destroying the harmonic property. Hence

$$AOA'A'' \barwedge BB''B'O.$$

Hence AB, $A'B'$, $OB,''$ OA'' envelop a conic and by Theorem IV the triangles AOB and $A'OB'$ are equivalent. Then in similar manner

$$BOC = B'OC' \quad \text{and} \quad COA = C'OA'.$$

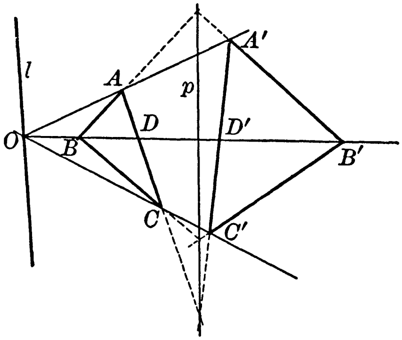
If the sign of the area be taken into consideration

$$-ABC = AOB + BOC + COA = A'OB' + B'OC' + C'OA' = -A'B'C'.$$

The theorem is therefore proved.

Theorem IX. If a triangle ABC is transformed into $A'B'C'$ by a projective reflection the areas of the triangles ABC and $A'B'C'$ are equal in magnitude and opposite in sign with reference to any line l drawn through the center of the reflection. (See figure 6.)

It is to be understood that the triangle ABC, and consequently the triangle $A'B'C'$, is that one of the four triangles ABC (into which the plane is divided by the lines AB, BC, CA, see p.35) which is not cut by the line l. It will be convenient to assume that the letter B has been assigned to the vertex so situated that the line OB cuts the base AC interiorly at D. Let OB cut $A'C'$ at D'. Inasmuch as AC is transformed into $A'C'$ and the line OB is left fixed by the reflection, it is evident that D and D' are corresponding points and are harmonically separated by O and p. Apply Theorem VI to the triangles into which ABC and $A'B'C'$ are divided by OBB'. By definition of equivalence with respect to l the triangles $D'AB'$ and $D'A'B'$ are equivalent. By Theorem VI the triangles ABD and $AD'B'$ are equivalent since O is one of the pairs of points which separate BB' and DD' harmonically. Hence



$$A'D'B' = AB'B' = ABD = -ADB;$$

and likewise

$$C'D'B' = CD'B' = CBD = -CDB;$$

Subtracting,

$$A'B'C' = -ABC.$$

The theorem is therefore proved.

Theorem X. The necessary and sufficient condition that a collineation which leaves the line l fixed shall transform any triangle ABC into a triangle $A'B'C'$ equivalent to ABC with regard to that line l is that the collineation be resoluble into the product of two projective reflections of which the poles lie upon the line l.

The sufficiency of the condition is immediately seen by aid of Theorem IX. For by the two reflections the area suffers merely two changes of sign and consequently is finally left unaltered. To discuss the necessity of the condition, suppose that the triangle ABC is transformed into an equivalent triangle $A'B'C'$. The lines AB, BC, CA cut out three points c, a, b on the line of reference l. The lines $A'B'$, $B'C'$, $C'A'$ cut out three points c', a', b'. Let T be the collineation which carries ABC into $A'B'C'$. Then

$$a[T]a', \qquad a[T]b', \qquad c[T]c'.$$

Let S be the linear transformation of the line l, not of the whole plane, which carries a, b, c into a', b', c'. S is known to be resoluble into two involutions.* Let these involutions be s_1, s_2; then

$$S = s_1 s_2; \qquad a,\ b,\ c\ [s_1 s_2]\ a',\ b',\ c'.$$

The involutions s_1, s_2, each consists in replacing the points of the line l by the harmonic points with respect to a pair of points p_1, O_1 and p_2, O_2. Draw through the points p_1 and p_2 lines p_1 and p_2 and consider the projective reflections t_1 and t_2 which have p_1, O_1, p_2 and O_2 respectively as axis and center. The product $[t_1 t_2]$ produces the same transformation on l as the product $[s_1\, s_2]$:

$$a,\ b,\ c\ [t_1 t_2]\ a',\ b',\ c'.$$

From the product of T and the inverse of $[t_1 t_2]$ which is $[t_2^{-1} t_1^{-1}]$ and apply the transformation to the line l; then

$$a,\ b,\ c\ [T]\ a',\ b',\ c'\ [t_2^{-1} t_1^{-1}]\ a,\ b,\ c,$$

that is,

$$a,\ b,\ c[T t_2^{-1} t_1^{-1}]\ a,\ b,\ c.$$

Let

$$T\, t_2^{-1} t_1^{-1} = R.$$

The transformation R carries the line l identically into itself. The triangle ABC is therefore carried into a triangle $A''B''C''$ such that the corresponding sides meet on l.

It remains to discuss the variations in R as the directions of the lines p_1 and p_2, which must necessarily pass through the points p_1 and p_2 on l, are varied. There are a number of cases to consider. These will be stated in a series of lemmas.

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* The general linear transformation in one dimension is resoluble in ∞^1 ways into the product of two involutions. See a paper on "Collineations of Space" by R. G. Wood, ANNALS OF MATHEMATICS, ser. 2, vol. 2(1900-01), p. 164, where further references are given.

Lemma 1. If the two centers O_1 and O_2 are distinct a given point P may be carried over by two reflections t_2 and t_1, whose axes pass through two fixed points p_1 and p_2 on l, into any assigned point P''.

For, by t_2 the point P may be carried into any point P' of the line PO_2. Then by t_1, P' may be carried into any point of the line joining $P'O_1$. This point evidently is any point P'' of the entire plane.

Lemma 2. In case the two centers O_1 and O_2 coincide and the two points p_1 and p_2 on l coincide, the point P may be carried over into any arbitrary point P'' by t_2t_1.

For the transformation upon l is in this case the identical transformation. Hence the common points O_1 and O_2, p_1 and p_2 may be chosen arbitrarily. Choose the points O_1 and O_2 as coincident with the point of intersection of the line PP'' with the line l. Choose also the points p_1 and p_2 at any arbitrary common position. Let the line p_2 be any line. Then P is carried into P'. Let the line p_1 be so chosen that P' is carried into P''. For this purpose p_1 and l must separate P' and P'' harmonically.

The case in which O_1 and O_2 coincide but the points p_1 and p_2 do not coincide need not be considered separately from the case of Lemma 1. For whether the points O or the points p be chosen as centers of the projective reflections is evidently quite immaterial as far as the displacement of the points upon the line l is concerned. Hence:

THEOREM XI. It is always possible to choose two projective reflections t_2 and t_1 which first produce any assigned transformation upon the line l joining their centers and secondly carry any preassigned point P into any other assigned point P''.

Consider next the transformation R. Let T carry the point A into A'. Choose t_2 and t_1 so that the product t_2t_1 carries A' into A. Then since t_2 and t_1 are involutory their inverse transformations t_2^{-1} and t^{-1} are equal respectively to t_2 and t_1, and the product $t_2^{-1}t_1^{-1}$ will carry A' into A. Hence

$$ABC\,[Tt_2^{-1}t_1^{-1}]\,AB''C'' \quad \text{or} \quad ABC\,[R]\,AB''C''.$$

It must be remembered that every transformation R leaves the line l identically fixed. Hence AB and AB'', AC and AC'' are collinear and BC and $B''C''$ intersect on l. But by supposition the areas ABC and $A'B'C'$ are equal. As the transformations t_2 and t_1 do not alter area with regard to their line of centers l, the areas ABC and $AB''C''$ are equal. Hence if $B''C''$ and

BC lie between the point A and the line l they are necessarily coincident and if they are separated by A and l they are necessarily divided harmonically by A and l.

Lemma 3. If the rôle of the points p_1 and O_1 upon the line l be interchanged—that is if the axis p_1 of the reflection be taken to pass through O_1 and the center of the reflection be taken at p_1 and if this transformation be designated by u to distinguish it from t_2, then if t_2t_1 carry a triangle ABC into $\bar{A}\bar{B}\bar{C}$ a transformation t_2u may be found which carries ABC into $\bar{A}\,\bar{B}'\bar{C}$ in such a manner that $\bar{B}$ and $\bar{B}'$, $\bar{C}$ and $\bar{C}'$ are harmonically separated by $\bar{A}'$ and l.

In the first place, by Lemmas 1 and 2 it is always possible to find a transformation t_2u which shall carry A into $\bar{A}$. Secondly, since t_2t_1 and t_2u produce the same transformation on l, the lines $\bar{A}\bar{B}\bar{B}'$ and $\bar{A}\bar{C}\bar{C}'$ are straight. Furthermore the lines $\bar{B}\bar{C}$ and $\bar{B}'\bar{C}'$ meet on l. The areas of the triangles $\bar{A}\bar{B}\bar{C}$ and $\bar{A}\bar{B}'\bar{C}'$ are each equal to that of ABC and hence are equal to each other. Therefore either $\bar{B}\,\bar{C}$ and $\bar{B}'\bar{C}'$ coincide or they are divided harmonically by the point A and the line l. Suppose they coincide. Then the transformations t_2t_1 and t_2u carry the triangle ABC into the triangle $\bar{A}\bar{B}\bar{C}$ and leave the line l fixed. The transformations are therefore equal. It must be remembered that t_1 and t_2 and u have been used for *any* projective reflections which have their axes passing through the points p_1, p_2, O, and their centers situated at O_1, O_2, p_1 respectively. The letter t_2 thus stands for a type of transformations rather than for any particular individual. Hence the t_2 in t_2t_1 need not be equal to the t_2 in t_2u. To designate this difference affix an accent to the second t_2. Then, on the supposition made above,

$$t_2t_1 = t'_2u.$$

Multiplying by t'_2 and remembering that the square of an involutory transformation is unity,

$$t'_2t_2t_1 = t'_2t'_2u = u.$$

Multiplying by t_1,

$$t'_2t_2 = ut_1.$$

The transformation ut_1 has the characteristic property that the axis of each of its factors passes through the center of the other. Let the axes intersect in some point Q. Then the transformation ut_1 is a projective reflection of which the center is Q and the axis l. For let P be any point of the plane. Let P be carried by u into P' and let P' be carried by t_1 into P''. Then the line QP is carried into QP''. But as the line l is identically fixed QPP'' is straight. Moreover by the harmonic property of the transformation t_1 it is evident that P and P'' are separated harmonically by Q and l. Therefore the transformation ut_1 is a projective reflection with Q as center and l as axis. Denote ut_1 by v; then

$$t_2' t_2 = ut_1 = v.$$

This equation states that the product of two transformations t_2 which have the same center O_2 and whose axes pass through the same point p_1 on l is an involutory transformation v — which is impossible, as a moment's consideration will render evident. Hence the supposition that $t_2 t_1$ and $t_2 u$ carried ABC into the same triangle $\bar{A}\bar{B}\bar{C}$ is impossible and the contrary supposition, which is the statement of the lemma, stands proved.

To turn back again to the consideration of the transformation R. It was seen that R carried the triangle ABC of Theorem X into a triangle $AB'C'$ which was either identical with ABC or the harmonic counterpart of it with regard to A and l. If this latter case obtains it is only necessary to replace $t_2 t_1$ by $t_2' u$ and the resulting transformation

$$R' = T t_2'^{-1} u^{-1}$$

carries ABC into ABC identically. In the former case R itself carries ABC into ABC identically. The transformations R and R' leave l fixed. Hence R or R', as the case may be, is the identical transformation. Let t be a transformation which may stand indifferently for t_2 or t_2' and let u stand likewise for t_1 or u as the case may require. Then the result of the investigations since the statement of Theorem X is

$$T t^{-1} u^{-1} = 1,$$

or

$$T = ut.$$

Hence Theorem X is proved. With the statement of this theorem in a little different form this paper will close: *The necessary and sufficient condition that a collineation of the plane shall be resoluble into two projective reflections is that the collineation shall leave areas unaltered when referred to one of the fixed lines of the collineation.*

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